SMP 16-19 Mathematics – Revision Notes
Unit 3 – Functions

Algebra Of Functions
1. Functions can be combined whereby $fg(x) = f(g(x))$ = g(x) followed by $f(x)$.
2. The set of values for which a function is defined is the domain (i.e. x values), and the set of values that the function can return is the range (i.e. y values).
3. Many-to-one functions have more than one value in the domain giving one value in the range. It is impossible to have many-to-one functions.
4. The inverse of a function is denoted by $f^{-1}(x)$, and is only a function if $f(x)$ is one-to-one.
5. The graphs of a function and its inverse function have reflection symmetry in the line $y = x$.
6. Parameters are values in a function that can vary, but for any given function mapping x onto y they will act as constants (e.g. a, b, and c in $y = ax^2 + bx + c$).
7. The image of $y = f(x)$ under a translation of $\begin{bmatrix} -p \\ q \end{bmatrix}$ is $y = f(x+p)+q$.
8. The image of $y = f(x)$ after reflection in the y-axis is $y = f(-x)$.
9. The image of $y = f(x)$ after reflection in the x-axis is $y = -f(x)$.
10. If $f(-x) = f(x)$ then f is an even function (i.e. is symmetric about the y-axis).
11. If $f(-x) = -f(x)$ then f is an odd function (i.e. has rotational symmetry about the origin).

Circular Functions
1. The sine and cosine functions are periodic – they repeat themselves after a period.
2. $y = \sin(x+c)° + d$ is obtained by a translation of $\begin{bmatrix} -c \\ d \end{bmatrix}$.
3. $y = a\sin x°$ is obtained by a one-way stretch parallel to the y-axis. a is the amplitude.
4. $y = \sin bx°$ is obtained by a one-way stretch parallel to the x-axis. $\left(\frac{360}{b}\right)$ is the period.
5. $y = \sin(bx+c)°$ is obtained by a stretch of $\frac{1}{b}$ followed by a phase shift of $\frac{-c}{b}$.
6. For $\sin a° = b$, then $a° = \sin^{-1} b$, i.e. the inverse function of sin. The principle values for this will be given by the range of the function – i.e. $-90° \le \sin^{-1} x \le 90°$.
7. For $\sin^{-1}$ and $\cos^{-1}$ the domain will be $\{x \in \mathbb{R} : -1 \le x \le 1\}$.
8. The trigonometric functions have an infinite number of solutions, and these can be found by looking at their periodic nature to find other solutions starting from the principle value.
9. $\tan x = \frac{\sin x}{\cos x} (\cos x \ne 0)$. Note that $\tan(-x) = -\tan x$.

Growth Functions
1. Growth is called exponential when there is a constant, called the growth factor, such that during each time interval the amount present is multiplied by this factor.
2. If the growth factor is less than 1, then exponential decay will occur.
3. The laws of indices: $a^0 = 1$, $a^{-m} = \frac{1}{a^m}$, $a^{\frac{1}{m}} = \sqrt[m]{a}$, $a^m \times a^n = a^{m+n}$, $a^m \div a^n = a^{m-n}$, $(a^m)^n = a^{mn}$
4. The general growth function has an equation of the form $y = Ka^x$ where K is the value of y when x=0, and a is the growth factor.
5. For $y = a^x$, $x = \log_a y$.
6. The laws of logs: $\log_a a = 1$, $\log_a 1 = 0$, $\log_a\left(\frac{1}{a}\right) = -1$, $\log_a a^x = a^{\log_a x} = x$, $\log_a mn = \log_a m + \log_a n$, $\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$
7. If $a > 0$ then $\log a^n = n\log a$. So for $a^x = b$, $x = \frac{\log b}{\log a}$.

Radians

1. π radians = 180°.
2. A radian is the angle subtended by an arc of unit length at the centre of a circle of unit radius.
3. For a sector of a circle, radius r, with angle θ radians:

$$\text{arc length } l = rq$$
$$\text{area } a = \tfrac{1}{2} r^2 q$$

4. With x in radians:

$$y = a \sin bx \Rightarrow \frac{dy}{dx} = ab \cos bx$$
$$y = a \cos bx \Rightarrow \frac{dy}{dx} = -ab \sin bx$$

The Constant e

1. $\frac{d}{dx}(e^x) = e^x$
2. $\frac{d}{dx}(e^{ax}) = ae^{ax}$
3. For natural logarithms (i.e. to base e), $\log_e x = \ln x$. This follows the same rules as the laws of logs.
4. $\frac{d}{dx}(\ln ax) = \frac{1}{x}$ and $\int \frac{1}{x}\,dx = \ln x$ for $x > 0$.
5. For the logistic curve, $y = \frac{A}{1 + Ke^{-lx}}$.

Transformations

1. Algebraic transformations:

Algebraic Transformation	Geometric Transformation
Replace x with kx	Stretch of factor $\frac{1}{y}$ from the y-axis
Replace y with ky	Stretch of factor $\frac{1}{x}$ from the x-axis
Replace x with $x + k$	Translation $\begin{bmatrix} -k \\ 0 \end{bmatrix}$
Replace y with $y + k$	Translation $\begin{bmatrix} 0 \\ -k \end{bmatrix}$
Replace x with $-x$	Reflection in y-axis
Replace y with $-y$	Reflection in x-axis
Interchange x and y	Reflection in $y = x$

2. For an ellipse of centre (p, q), of major axis $2a$ and minor axis $2b$, then the equation will be:

$$\left(\tfrac{1}{a}(x-p)\right)^2 + \left(\tfrac{1}{b}(y-q)\right)^2 = 1$$